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Construction of Sturmian sequences

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Abstract
It is well known that a Sturmian sequence can be regarded as a rotation sequence or a balanced sequence. In this paper, by a rotation sequence, we first construct a series of sequences with complexity $kn + 1$, then, from these sequences, we reconstruct other Sturmian sequences and discuss their relationships.

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1. Introduction

A coarse-grained description of a dynamical system can be represented by an infinite symbolic sequence (the dynamical system concerned here is an iterative system generated by a continuous map from a compact topological space into itself), see [14]. The symbolic sequences obtained in this way, as the simplest dynamics with respect to the shift operator, fits well the framework of formal language. Therefore, symbolic sequences may be studied from the viewpoint of language and grammar complexity. Given an infinite sequence F, its language complexity $p(n, F)$ is defined to be the number of its factors of length n, which has been extensively studied in the last few years, e.g. [21]. From the definition it is easily seen that F is periodic if $p(n, F)$ is bounded. For the sequence F, a rich and instructive task is to compute its complexity function $p(n, F)$; however, it is generally not easily accessible when F is aperiodic. So Sturmian sequences, the aperiodic infinite symbolic sequences with minimal language complexity $(p(n, F) = n + 1)$, have attracted a lot of attention in many fields of mathematics, physics and biology [3, 21]. After the discovery of quasicrystals Sturmian dynamical systems became particularly attractive to mathematical physicists [1, 7, 12, 16, 17, 24, 25], as they can serve as simple models for one-dimensional quasicrystals. So the study of Sturmian sequences plays an important role in the study of quasicrystal structure. Sturmian sequences admit various equivalent definitions in different manners such as rotation sequences, cutting sequences, Christoffel words, balanced sequences, and so on.

This paper is motivated by the following idea: since a numerical orbit in a dynamical system can be reduced to a symbolic sequence, could we reduce a symbolic sequence with a higher language complexity to one of lower language complexity through some coarse-grained processes? In this paper we shall give a positive answer to this question. We first review, in the next section, the terminology on rotation sequences, based on which we construct a class of sequences F_k with $p(n, F_k) = kn + 1$ $(k = 1, 2, \ldots)$ from a Sturmian sequence. In section 3, performing a coarse-grained description for F_k, we get other Sturmian sequences, and discuss their relations.

2. From a Sturmian sequence to F_k with $p(n, F_k) = kn + 1$

Before starting our discussion we briefly review the terminology on words. Let $A = \{l_1, l_2, \ldots, l_k\}$ be an alphabet with k letters. A finite (infinite) string w is called a word (sequence) if $w = w_1 w_2 \cdots w_n (w = w_1 w_2 \cdots w_n \cdots)$ with $w_i \in A$. We denote by A^* the set of all words over A. If $w = uv$ is the concatenation of two words $u = u_1 u_2 \cdots u_r$, $v = v_1 v_2 \cdots v_s$, then w is defined as $u_1 u_2 \cdots u_r v_1 v_2 \cdots v_s$. We denote by u^n the concatenation of n copies of u. The concatenation of a word and a sequence can be defined similarly. Let $w = w_1 w_2 \cdots w_n$. We call n the length of w denoted by $|w|$. A word u is called a factor (resp. a prefix, resp. a suffix) of a word w if there exist words x, y such that $w = xuy$ (resp. $w = uy$. resp. $w = xu$). In this case, we say $(|x|, u)$ is an occurrence of u in w. The factor (resp. a prefix, resp. a suffix) is proper if $xy \neq \varepsilon$ (resp. $y \neq \varepsilon$. resp. $x \neq \varepsilon$), where ε is the empty word. The language of length n of a sequence F, denoted by $\Omega_n(F)$, is the set of all factors of F of length n. $\Omega(F) = \cup_{n\geqslant 0}\Omega_n(F)$ is the set of all factors of F. The complexity function of F is defined as $p(n, F) = \#\Omega_n(F)$. A sequence F is called Sturmian if $p(n, F) = n + 1$. Throughout this paper, we assume $A = \{a, b\}$, an alphabet with two letters.

We now review the rotation sequences. Let $T = [0, 1)$. Consider a map from T into itself $f : T \to T$ defined by

$$f(x) = x + \alpha \pmod 1,$$

where x is a real number and α is an irrational number. The iteration $f^n : T \to T$ is defined inductively by $f^0(x) = x$, $f^{n+1}(x) = f(f^n(x))$. It is clear that $f^n(x) \neq x$ for all $n > 0$. Starting the initial point $x_0 = \beta$, we have a numerical orbit

$$x_0, x_1, \ldots, x_n, \ldots \tag{1}$$

by iterating the map: $x_1 = f(x_0)$ and $x_n = f(x_{n-1})$ for $n > 1$. Considering two intervals $I_0 = [0, \alpha)$ and $I_1 = [\alpha, 1)$ on T, we denote by μ_α the coding function defined by

$$\mu_\alpha(x) = \begin{cases} a & \text{if} \quad x \in I_0 \\ b & \text{if} \quad x \in I_1. \end{cases}$$

Then a coarse-grained description of the numerical orbit is realized by the following symbolic sequence:

$$\mu_\alpha(x_0), \mu_\alpha(x_1), \ldots, \mu_\alpha(x_n), \ldots. \tag{2}$$

We call this sequence, denoted by $F_{\alpha,\beta}$, a rotation sequence defined by α and β. From [22] we get that rotation sequences and Sturmian sequences are equivalent. We state this result as follows:

Theorem 2.1 [22]. *Every rotation sequence is a Sturmian sequence. Conversely, for each Sturmian sequence there are real numbers α and β such that $F_{\alpha,\beta}$ is just that Sturmian sequence.*

Let α be an irrational number and β be a real number and let

$$F_{\alpha,\beta} = u_0u_1\cdots u_n\ldots, \qquad \text{where} \quad u_i \in \{a, b\}.$$

From theorem 2.1 we know that $p(n, F_{\alpha,\beta}) = n+1$. Let $S_k = \{s_1, s_2, \ldots, s_{k+1}\}$ denote the set of factors of $F_{\alpha,\beta}$ of length k. Then $F_{\alpha,\beta}$ can be written as the product of the elements in S_k, denoted by $F_k = v_0v_1\cdots v_n\cdots$, where $v_i \in S_k$.

Theorem 2.2. *The complexity function of F_k on S_k is $p(n, F_k) = kn + 1$.*

To prove this theorem, we first give a few lemmas.

Lemma 2.3 [13]**.** *A Sturmian sequence is recurrent, that is, every word that occurs in the sequence occurs an infinite number of times.*

According to the fact that irrational rotations of a circle are minimal as topological dynamical systems, we have the following lemma:

Lemma 2.4 [27]**.** *Let α be an irrational number and $(p, q) \subset [0, 1)$ with $p < q$. Then there exists an integer $n > 0$ such that $x = n\alpha \pmod 1 \in (p, q)$.*

Lemma 2.5. *Let k be a fixed positive integer, $F_{\alpha,\beta} = u_0u_1\cdots u_n\ldots$, $w_i^{(k)} = u_{ik}u_{ik+1}\cdots u_{(i+1)k-1}$ for $i \geqslant 0$ and let S_k' denote the set of distinct $w_i^{(k)}$. Then $S_k' = S_k$, and therefore $|S_k'| = k+1$.*

Proof. It is clear that $S_k' \subseteq S_k$. Now we prove that $S_k \subseteq S_k'$. Suppose $t = t_1t_2\cdots t_k \in S_k$, where $t_i \in A$. We have known that $F_{\alpha,\beta}$ is generated by the numerical orbit $x_0x_1\cdots x_n\cdots$, where $x_1 = f(x_0)$ and $x_n = f(x_{n-1})$ for $n > 1$. By theorem 2.1, there exists $m \geqslant 0$ such that $t_i = \mu_\alpha(x_{m+i-1})$ and $\min\{|x_i|, |x_i - \alpha|, |x_i - 1|\} > 0$ for $0 < i < k+1$ since $f^n(x) \neq x$ for any n. Let $y = \min\{|x_i|, |x_i - \alpha|, |x_i - 1|$ with $0 < i < k+1\}$. From lemma 2.4, there exists $j > 0$ such that $x_{jk} = jk\alpha \pmod 1 \in (x_m - y, x_m + y) \subset [0, 1)$. Then we get that $t_i = \mu_\alpha(x_{jk+i-1})$ for $0 < i < k+1$, which implies that $t = t_1t_2\cdots t_k = x_{jk}x_{jk+1}\cdots x_{(j+1)k-1} = w_j^{(k)} \in S_k'$, which implies that $|S_k'| = |S_k| = k+1$. □

Proof of theorem 2.2. Write $F_k = w_0^{(k)}w_1^{(k)}\cdots w_n^{(k)}\cdots$, where $w_i^{(k)} \in S_k$. We need to prove $p(n, F_k) = kn+1$ on S_k. Since $p(kn, F_{\alpha,\beta}) = kn+1$, we have that $p(n, F_k) \leqslant kn+1$. Then we need only to prove that $p(n, F_k) \geqslant kn+1$. Let $F_k(n)$ be the set of all distinct factors of length n of the sequence F_k on S_k. For any $t \in S_{kn}'$, there exists j such that $t = w_j^{(kn)}$. According to lemma 2.5, it is clear that $w_j^{(kn)} = w_{nj}^{(k)}w_{nj+1}^{(k)}\cdots w_{nj+n-1}^{(k)} \in F_k(n)$, which implies that $p(n, F_k) = |F_k(n)| \geqslant |S_{kn}'| = kn+1$. □

Example. Let F be the Fibonacci sequence, which can be generated by following: $F(0) = a$, $F(-1) = b$, $F(n) = F(n-1)F(n-2)$, so

$$F = F(\infty) = abaababaabaababaababaabaababaabaababaababaabaababaababa\cdots.$$

It is a well-known example of Sturmian sequences [21].

Let $\pi_k : S_k \longrightarrow [k+1]$ be a one-to-one mapping, where $[k+1] = \{0, 1, \ldots, k\}$. We define $\pi_k(F) = \pi_k\big(w_0^{(k)}\big)\pi_k\big(w_1^{(k)}\big)\cdots\pi_k\big(w_n^{(k)}\big)\cdots$. So we get that $F_k = \pi_k(F)$ on $[k+1]$. Then we have

$$F_2 = 0122012200120012001220122001200120012201220122\cdots$$

$$F_3 = 0011230001123000123300012330001233001123300112\cdots$$

$$F_4 = 010123232342304040101012323434340401010123232\,3\cdots$$

$$F_5 = 001123444550011234444550011233445500112334455 00\cdots$$

$$F_6 = 012034503603250460125345012034504603250460125 3\cdots.$$

3. From F_k to Sturmian sequences

Now, we construct a series of Sturmian sequences from F_k. We first recall another equivalent definition of Sturmian sequences, that is, balanced sequences.

If U is a finite word over the alphabet A, we denote $|U|_a$ the number of occurrences of the letter a in U. We say U is even if $|U|_a$ is even, otherwise U is odd. A sequence u over A is balanced if, for any pair of words U, V of the same length occurring in u, we have $||U|_a - |V|_a| \leqslant 1$. Reference [22] says that balanced sequences and Sturmian sequences are equivalent.

Theorem 3.1 [22]**.** *A sequence u is Sturmian if and only if it is a noneventually periodic balanced sequence over two letters.*

From theorem 3.1 we know that for any pair of words U and V, which are two factors of a Sturmian sequence, we have $|U|_a = |V|_a$ if U and V have the same length and parity; otherwise, $||U|_a - |V|_a| = 1$. Let $F_{\alpha,\beta} = u_0u_1\cdots u_n\cdots$ be a Sturmian sequence and $F_k = v_0v_1\cdots v_n\cdots$ denote the sequence on S_k. Let $\delta : S_k \longrightarrow \{a, b\}$ defined as: for any $v \in S_k$, $\delta(v) = a$ if v is even, otherwise $\delta(v) = b$. Then we have the following theorem:

Theorem 3.2. *Let $F_k' = \delta(F_k) = \delta(v_0)\delta(v_1)\cdots\delta(v_n)\cdots$. Then F_k' is a Sturmian sequence.*

Proof. Assume that F_k' is not Sturmian. From theorem 3.1 it follows that there exist two factors U' and V' of F_k' with the same length m, such that $||U'|_a - |V'|_a| > 1$. Let n_{e} (resp. n_{o}) denote the number of a occurring in $v \in S_k$ with v being even (resp. odd). It is not difficult to find that $|n_{\mathrm{e}} - n_{\mathrm{o}}| = 1$. Since $F_k' = \delta(F_k)$, there exist two factors U, V of F_k such that $\delta(U) = U'$ and $\delta(V) = V'$. Then we have

$$\begin{aligned}
||U|_a - |V|_a| &= |(n_{\mathrm{e}}|U'|_a + n_{\mathrm{o}}(|U'| - |U'|_a)) - (n_{\mathrm{e}}|V'|_a + n_{\mathrm{o}}(|V'| - |V'|_a))| \\
&= |n_{\mathrm{e}}(|U'|_a - |V'|_a) + n_{\mathrm{o}}(|U'| - |U'|_a - |V'| + |V'|_a)| \\
&= |n_{\mathrm{e}}(|U'|_a - |V'|_a) - n_{\mathrm{o}}(|U'|_a - |V'|_a)| \\
&= |(n_{\mathrm{e}} - n_{\mathrm{o}})(|U'|_a - |V'|_a)| \\
&= ||U'|_a - |V'|_a| > 1,
\end{aligned}$$

which contradicts that $F_{\alpha,\beta}$ is a Sturmian sequence. □

Example. Let F be the Fibonacci sequence. By theorem 3.2 we obtain a series of Sturmian sequences as follows:

$$\begin{aligned}
F_2' = {} & babbbabbbbabbbabbbabbbabbbabbbbabbbabbbabbbabbbabbbabbbbabbbabbbab \\
& bbabbbbabbbabbbabbbabbbbabbbabbbabbbabbbbabbbabbbabbbabbbbabbb \\
& abbbbabbbabbbabbbabbbbabbbabbbabbbabbbbabbbab\cdots; \\
F_3' = {} & aaaabaaaaaabaaaaabaaaaaabaaaaaabaaaaaabaaaaaabaaaaaa \\
& baaaaaabaaaaabaaaaaabaaaaaabaaaaaabaaaaaabaaaaaabaaaaaa \\
& baaaaabaaaaaabaaaaaabaaaaaabaaaaaabaaaaaabaaaaaabaaaa\cdots;
\end{aligned}$$

$F'_4 =$ *babaababababababababaababababababababaababababababababaabababababababa*
babaababababababababaababababababababaababababababababaabababababababa
baababababababababaababababababababaababababababababaababababababab $\cdots$;
$F'_5 =$ *bbbbabbbbbbbbbbabbbbbbbbbbabbbbbbbbbbabbbbbbbbbbabbbbbbbbbbabbbbbbbbbbab*
bbbbbbbbbabbbbbbbbbbabbbbbbbbbbabbbbbbbbbbabbbbbbbbbbabbbbbbbbbbabbbbbb
bbbbabbbbbbbbbbabbbbbbbbbbabbbbbbbbbbabbbbbbbbbbabbbbbbbbbbabbbbbbbbbbab
bbbbbbbbbabbbbbbbbb $\cdots$;
$F'_6 =$ *aabaabaabaaabaabaaabaabaaabaabaabaaabaabaaabaabaaabaab*
aabaaabaabaaabaabaaabaabaabaaabaabaaabaabaaabaabaabaaa
baabaaabaabaaabaabaabaaabaabaaabaabaaabaabaabaaabaab $\cdots$.

Now, we consider the relationship of $F_{\alpha,\beta}$ and F'_k. We have the following theorem.

Theorem 3.3. *$F'_k = F_{\alpha_k,\beta_k}$, where $\beta_k = \beta + (k-1)\alpha \pmod 1$ and $\alpha_k = k\alpha \pmod 1$.*

We first prove two lemmas. Let $F_{\alpha,\beta} = \mu_\alpha(x_0)\mu_\alpha(x_1)\cdots\mu_\alpha(x_n)\cdots$, where $x_0 = \beta$ and $x_n = f(x_{n-1})$ for $n \geqslant 1$. $I_0^k = [0, \alpha_k)$ and $I_1^k = [\alpha_k, 1)$ are two intervals on T. Let $\Delta_k(x_i) = \mu_\alpha(x_{i-k+1})\mu_\alpha(x_{i-k+2})\cdots\mu_\alpha(x_i)$ for $i \geqslant k-1$. It is clear that $\Delta_{k+t}(x_i) = \Delta_k(x_{i-t})\Delta_t(x_i)$.

Lemma 3.4. *If $x_i \in I_0^k$ and $x_j \in I_1^k$, then $\Delta_k(x_i)$ and $\Delta_k(x_j)$ have different parities.*

Proof. We apply induction on k. When $k = 1$, the lemma is trivially satisfied. Now assume it is correct for $k = n$. Without loss of generality, we assume that $\Delta_n(x)$ is odd if $x \in I_0^n$; otherwise $\Delta_n(x)$ is even. Then we prove that the lemma is correct for $k = n + 1$.

Suppose $\alpha > 1/2$. Then $\alpha > \alpha_2$. There are three cases to be considered.

Case 1. $\alpha_{n+1} < \alpha_2$. Then we have $\alpha_n < \alpha$. If $x \in [0, \alpha_{n+1}) = I_0^{n+1}$, then $x - \alpha \pmod 1 \in [1-\alpha, \alpha_n) \subset I_0^n$. So $\Delta_n(x - \alpha \pmod 1)$ is odd and $\Delta_{n+1}(x) = \Delta_n(x - \alpha \pmod 1)\Delta_1(x)$ is even. If $x \in [\alpha_{n+1}, \alpha)$, then $x - \alpha \pmod 1 \in [\alpha_n, 1) \subset I_1^n$, which implies that $\Delta_n(x - \alpha \pmod 1)$ is even and $\Delta_{n+1}(x) = \Delta_n(x - \alpha \pmod 1)\Delta_1(x)$ is odd. When $x \in [\alpha, 1)$, it is clear that $x - \alpha \pmod 1 \in [0, 1-\alpha)$, which leads that $\Delta_n(x - \alpha \pmod 1)$ is odd and $\Delta_1(x)$ is even. So $\Delta_{n+1}(x)$ is odd. Then we have $\Delta_{n+1}(x)$ is odd if $x \in I_1^{n+1}$.

Case 2. $\alpha > \alpha_{n+1} > \alpha_2$. Then $\alpha < \alpha_n < 1$ and we have

$$x - \alpha \pmod 1 \in \begin{cases} [1-\alpha, \alpha_n) & \text{if} \quad x \in [0, \alpha_{n+1}), \\ [\alpha_n, 1) & \text{if} \quad x \in [\alpha_{n+1}, \alpha), \\ [0, 1-\alpha) & \text{if} \quad x \in [\alpha, 1), \end{cases}$$

from which it follows that $\Delta_{n+1}(x)$ is even if $x \in I_0^{n+1}$; otherwise $\Delta_{n+1}(x)$ is odd.

Case 3. $\alpha_{n+1} > \alpha$. Then have $0 < \alpha_n < 1 - \alpha$ and

$$x - \alpha \pmod 1 \in \begin{cases} [1-\alpha, 1) & \text{if} \quad x \in [0, \alpha), \\ [0, \alpha_n) & \text{if} \quad x \in [\alpha, \alpha_n), \\ [\alpha_n, 1-\alpha) & \text{if} \quad x \in [\alpha_{n+1}, 1), \end{cases}$$

which implies that $\Delta_{n+1}(x)$ is odd if $x \in I_0^{n+1}$; otherwise $\Delta_{n+1}(x)$ is even. Then the lemma is correct for $\alpha > 1/2$.

When $\alpha < 1/2$, the proof is similar to the case of $\alpha > 1/2$, we omit it.

The proof is complete. □

Lemma 3.5. *We have that x_i and x_j are in same interval of I_0^k and I_1^k if $\Delta_k(x_i)$ and $\Delta_k(x_j)$ have same parity; otherwise, x_i and x_j are in different intervals.*

Proof. From lemma 3.4, it is not difficult to prove that $\Delta_k(x_i)$ and $\Delta_k(x_j)$ have same parities if and only if x_i and x_j are in the same interval, which implies, by reduction to absurdity, that x_i and x_j are in different intervals if $\Delta_k(x_i)$ and $\Delta_k(x_j)$ have different parity. The proof is complete. □

Proof of theorem 3.3. Let $\beta_k = x_0 + (k-1)\alpha \,(\mathrm{mod}\, 1)$. Then we have $F_{\alpha_k,\beta_k} = \mu_{\alpha_k}(x_{k-1})\mu_{\alpha_k}(x_{2k-1})\cdots\mu_{\alpha_k}(x_{nk-1})\cdots$. From lemmas 3.4 and 3.5, we conclude that $F'_k = F_{\alpha_k,\beta_k}$. □

Remark 3.6. There is much current research on discrete one-dimensional Schrödinger operators in $\ell^2(\mathbb{Z})$ with Sturmian potentials, namely,

$$(H_{\lambda,\alpha,\beta}u)(n) = u(n+1) + u(n-1) + \lambda F_{\alpha,\beta}u(n), \tag{3}$$

where $\alpha \in (0, 1)$ is irrational, $\beta \in [0, 1)$ and $\lambda \neq 0$, along with the corresponding difference equation

$$H_{\lambda,\alpha,\beta}u = Eu. \tag{4}$$

The operator family (3) describes a standard one-dimensional quasicrystal model [20, 24] and has been studied in many papers [1, 2, 6–9, 15, 17, 18, 23]. From (3) it is seen that different Sturmian sequences determine different Schrödinger operators. Therefore, to know the relationship of the structures of quasicrystals described by Schrödinger operators it is essential for us to know the relationship of different Sturmian sequences. From theorem 3.2 we know that by a Sturmian sequence $F_{\alpha,\beta}$ we can construct a series of Sturmian sequences F_{α_k,β_k} where k is a positive integer and α_k, β_k are given in theorem 3.3. Moreover, following the construction process given in sections 2 and 3, by $F_{\alpha,\beta}$ we can write the sequence F_{α_k,β_k} term by term. Hence, from $H_{\lambda,\alpha,\beta}u$ we can immediately obtain $H_{\lambda,\alpha_k,\beta_k}u$.

4. Conclusion

In this paper, we use a Sturmian sequence to form a series of sequences with higher language complexity, by which we then construct other Sturmian sequences. This progress answers a question: a sequence with higher language complexity can be reduced to a sequence with lower language complexity by some coarse-grained methods. These results can be applied to the study of quasicrystal structure as pointed in remark 3.6.

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References

[1] Bellissard J, Iochum B, Scoppola E and Testard D 1989 Spectral properties of one-dimensional quasi-crystals *Commun. Math. Phys.* **125** 527–43

[2] Bellissard J, Iochum B and Testard D 1991 Continuity properties of the electronic spectrum of 1D quasicrystals *Commun. Math. Phys.* **141** 353–80

[3] Berstel J 1996 Recent results in Sturmian words *Developments in Language Theory* ed J Dassow and A Salomma (Singapore: World Scientific) pp 13–24

[4] Brezinski C 2001 Dynamical systems and sequence transformations *J. Phys. A: Math. Gen.* **34** 10659–69

[5] Cao W T and Wen Z Y 2003 Some properties of the factors of Sturmian sequence *Theo. Comput. Sci.* **304** 365–85

[6] Damanik D 2004 Aversion of Gordon's theorem for multi-dimensional Schrödinger operators *Trans. Am. Math. Soc.* **356** 495–507

[7] Damanik D 2000 Gordon-type arguments in the spectral theory of one-dimensional quasicrystals *Directions in Mathematical Quasicrystals (CRM Monographs, Series 13)* (Providence, RI: American Mathematical Society) pp 277–305

[8] Damanik D 1998 α-continuity properties of one-dimensional quasicrystals *Commun. Math. Phys.* **192** 169–82

[9] Damanik D and Lenz D 1999 Uniform specral properties of one-dimensional quasicrystals: I. Absence of eigenvalues *Commun. Math. Phys.* **207** 687–96

[10] Damanik D and Lenz D 2003 Powers in Sturmian sequence *Eur. J. Comb.* **24** 377–90

[11] Damanif D and Lenz D 2002 The index of sturmian sequence *Eur. J. Comb.* **23** 23–9

[12] de Bruijn N G 1981 Sequences of zeros and ones generated by special production rules *Indag. Math.* **43** 27–37

[13] Pytheas Fogg N 2002 *Substitutions in Dynamics, Arithemetics and Combinatorics* (Berlin: Springer)

[14] Hao B L and Zheng W M 1998 *Applied Symbolic Dynamics and Chaos* (Singapore: World Scientific)

[15] Iochum B, Raymond L and Testard D 1992 Resistance of one-dimensional quasicrystals *Physica* A **187** 353–68

[16] Iochum B and Testard D 1991 Power law growth for the resistance in the Fibonacci model *J. Stat. Phys.* **65** 715–23

[17] Kaminaga M 1996 Absence of point spectrum for a class of discrete Schrödinger operators with quasiperiodic potential *Forum Math.* **8** 63–9

[18] Kohmoto M, Kadanoff Leo P and Tang C 1983 Localization problem in one dimension: mapping and escape *Phys. Rev. Lett.* **50** 1870–2

[19] Krasikov I, Rodgers G J and Tripp C E 2004 Growing random sequences *J. Phys. A: Math. Gen.* **37** 2365–70

[20] Luck J and Petritis D 1986 Phonon spectra in one-dimensional quasicrystals *J. Stat. Phys.* **42** 289–310

[21] Lothaire M 2002 *Combinatorics on Words* (Cambridge: Cambridge University Press)

[22] Morse M and Hedlund G A 1940 Symbolic dynamics: II. Sturmian trajectories *Am. J. Math.* **62** 1–42

[23] Ostlund S, Pandit R, Rand D, Schellnhuber H J and Siggia E D 1983 One-dimensional Schrödinger equation with an almost periodic potential *Phys. Rev. Lett.* **50** 1873–6

[24] Senechal M 1995 *Quasicrystals and Geometry* (Cambridge: Cambridge University Press)

[25] Sütö A 1987 The spectrum of a quasiperiodic Schrödinger operator *Commun. Math. Phys.* **111** 409–15

[26] Tan B and Wen Z Y 2003 Invertible substitutions and Sturmian sequences *Eur. J. Comb.* **24** 983–1002

[27] Walters P 1975 *Ergodic Theory* (New York: Springer)

[28] Wolfram S 1984 Computation theory of the general quartic map *Commun. Theor. Phys.* **22** 43